MATH 301   INTRO TO ANALYSIS   FALL 2016

Homework 02

Professional Problem

Recall that **I** stands for the set of irrational numbers.

(a) Show that if $a,\ b \in \mathbf{Q}$ then so are ab and $a+b$.

(b) Show that if $a \in \mathbf{Q}$, $a \neq 0$ and $t \in \mathbf{I}$ then $at \in \mathbf{I}$ and $a+t \in \mathbf{I}$.

(c) Part (a) can be said by ***Q*** *is closed under addition and multiplication*. Is **I** closed under addition or multiplication? Justify your answer. (Abbott, 1.4.1)

Written Problems

1. Let $A \subseteq \mathbf{R}$ be nonempty and bounded above, and let $c \in \mathbf{R}$. Define the set $cA = \{ca :\ a \in A\}$.
   (a) If $c \geq 0$, show that $\sup(cA) = c\sup A$.
   (b) What happens if $c < 0$? Make an analogous statement and prove it.
2. (a) If $\sup A < \sup B$, show that there exists an element $b \in B$ that is an upper bound for A.
   (b) Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.
3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$.
4. Let $a < b$ be real numbers and consider the set $T = \mathbf{Q} \cap [a, b]$. Show that $\sup T = b$.
5. Which of the following sets are dense in **R**? Take $p \in \mathbf{Z}$ and $q \in \mathbf{N}$ in every case.
   (a) $\{p/q :\ q \leq 10\}$.
   (b) $\{p/q :\ q$ is a power of $2\}$.
   (c) $\{p/q :\ 10|p| \geq q\}$.
6. (a) Show that $(a, b) \sim \mathbf{R}$ for any interval (a, b).
   (b) Show that an unbounded interval like $(a, \infty) = \{x :\ x > a\}$ has the same cardinality as **R** as well.
   (c) Show that $[0, 1) \sim (0, 1)$ by exhibiting a one-to-one onto function between them.
7. (a) Why is $A \sim A$ for every set A?
   (b) Given sets A and B, explain why $A \sim B$ if and only if $B \sim A$.
   (c) Given sets A, B and C, show that if $A \sim B$ and $B \sim C$ then $A \sim C$.
   (Hence, $\sim$ is an equivalence relation by (a)–(c).)

8. Let $S = \{(x, y) : 0 < x < 1, 0 < y < 1\}$.
   (a) Find a one-to-one function that maps $(0, 1)$ *into*, but not necessarily onto, S.
   (b) Use the fact that every number has a decimal expansion to produce a one-to-one function that maps S into $(0, 1)$. Discuss whether the formulated function is onto.
   (c) How would you use your results to argue that $S \sim (0, 1)$?

Solutions

Professional Problem

(a) Let $a = m/n$ and $b = p/q$, $m, n, p, q \in \mathbf{Z}$ and $n, q \neq 0$. Then

$$a + b = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq} \text{ and } ab = \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$$

are both rational because $mq + np$, mp, $nq \in \mathbf{Z}$ and $nq \neq 0$.

(b) $at \in I$: Assume the opposite, i.e. $at \in \mathbf{Q}$. Then $a = m/n$ and $at = p/q$ with $m, n, p, q \in \mathbf{Z}$ and $n, q \neq 0$. Also $m \neq 0$ because $a \neq 0$. Solve for t:

$$t = \frac{at}{a} = \frac{\frac{p}{q}}{\frac{m}{n}} = \frac{np}{mq}$$

which is rational. This is a contradiction.

$a + t \in I$: Assume the opposite, i.e. $a + t \in \mathbf{Q}$. Then $a = m/n$ and $a + t = p/q$ with $m, n, p, q \in \mathbf{Z}$ and $n, q \neq 0$. Solve for t:

$$t = (a + t) - a = \frac{p}{q} - \frac{m}{n} = \frac{np - mq}{nq}$$

which is rational. This is a contradiction.

(c) I is NOT closed under addition or multiplication.
Examples: $\sqrt{2}$ and $-\sqrt{2}$ are both irrational but $\sqrt{2} + (\sqrt{2}) = 0 \in \mathbf{Q}$ and $\sqrt{2} \cdot (-\sqrt{2}) = 2 \in \mathbf{Q}$.

Written Problems

1. (a) Denote the least upper bound by $s = \sup A$. We wish to show that $\sup(cA) = cs$.
(Proof that cs is an upper bound). If $s = \sup A$ then for all $a \in A$, $a \leq s$. Since $c \geq 0$, $c \cdot a \leq c \cdot s$. So cs is an upper bound of cA.
(Proof that cs is *least*). Assume that cs is NOT the least upper bound of cA, then cA has another upper bound b, $b < cs$. Since b is an upper bound of cA, $b \geq ca$ for all $a \in A$, so $b/c \geq a$ for all a, which means b/c is an upper bound for A. This is a contradiction because $b/c < s$, which makes b/c an upper bound that is less than s, the *least* upper bound.

(b) Denote the greatest upper bound by $t = \inf A$. We wish to show that $\sup(cA) = ct$.
(Proof that ct is an upper bound). If $t = \inf A$ then for all $a \in A$, $a \geq t$. Since $c \leq 0$, $c \cdot a \leq c \cdot t$. So ct is an upper bound of cA.
(Proof that ct is *least*). Assume that ct is NOT the least upper bound of cA, then cA has another upper bound b, $b < ct$. Since b is an upper bound of cA, $b \geq ca$ for all $a \in A$, so $b/c \leq a$ for all a, which means b/c is a lower bound for A. This is a contradiction because $b/c > t$, which makes b/c a lower bound that is greater than t, the *greatest* lower bound.

2. Let $x = \sup A$ and $y = \sup B$.

   (a) Consider $y > x$. By Lemma 1.3.8, for $\epsilon = \dfrac{y-x}{2}$ there exists $b \in B$ so that $b > y - \epsilon = \dfrac{x+y}{2}$. Note that $y > \dfrac{x+y}{2} > x$. So $b > x = \sup A$. Therefore b is an upper bound of A.

   (b) Consider $y \geq x$. When $y = x$ we do not have the same conclusion as in (a). Counterexample: $A = B = [0, 1]$.

3. It suffices to prove that *this set has no element.* Assume the opposite, that there is a number $x \in \bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$. Then $x \in (0, 1/n)$ for all $n \in \mathbf{N}$. This means $x < 1/n$ for all $n \in \mathbf{N}$. This is a contradiction to the Archimedean Property which states that for any positive number x there is a positive integer n such that $x > 1/n$.

4. It is obvious that b is an upper bound. To show it is the *least*, consider any $\epsilon > 0$. By the density of rational numbers, there is always a rational number r between $b - \epsilon$ and b. Since $r \in T$, by Lemma 1.3.8, $b = \sup T$.

5. (a) Not dense.

   (b) Dense.

   (c) $\frac{|p|}{q} \geq \frac{1}{10}$: not dense.

6. (a) Consider $f(x) = \dfrac{x - \frac{a+b}{2}}{(x-a)(x-b)}$. (It suffices to demonstrate the graph and refer to horizontal line test.)

   (b) It suffices to show that $(a, \infty) \sim (0, 1)$ because $(0, 1) \sim \mathbf{R}$. Proof: consider $f : (a, \infty) \to (0, 1)$, $f(x) = \dfrac{1}{x - a + 1}$.

   (c) The easiest proof is to show that they are 1-to-1 *into* each other. $[0, 1) \to (0, 1)$: $f(x) = 0.1 + \frac{x}{2}$ is 1-to-1. $(0, 10 \to [0, 1)$: $g(x) = x$ is 1-to-1. So $[0, 1) \sim (0, 1)$

7. (a) Consider $f : A \to A$, $f(x) = x$, is bijective.

   (b) If $A \sim B$ then there is a bijection $f : A \to B$. Since f is bijective, it has an inverse function f^{-1} which is a bijection from B to A.

   (c) If $A \sim B$ and $B \sim C$ then there are bijections $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ is also a bijection. Therefore $A \sim C$.

8. (a) $f(x) = (x, 0.5)$, $f : (0, 1) \to S$, is 1-to-1.

(b) Consider the decimal expressions $x = (0.x_1x_2x_3x_4\cdots)$ and $y = (0.y_1y_2y_3y_4\cdots)$. Let function $g : S \to (0,1)$ be defined by

$$g(x, y) = (0.x_1y_1x_2y_2x_3y_3\cdots).$$

Then g is 1-to-1.

(c) $(0,1) \sim S$ because each one is 1-to-1 *into* the other.